HOMOGENEOUS REAL HYPERSURFACES OF TYPES (A) AND (B) IN A NONFLAT COMPLEX SPACE FORM

KATSUFUMI YAMASHITA

This doctoral thesis consists of two parts. In the first part we characterize the homogeneous real hypersurface of type (B) with two distinct constant principal curvatures in $\mathbb{C}H^n(c)$ ($n \geqq 2$). In the second part we characterize homogeneous real hypersurfaces of type (A) in a nonflat complex space form.

The author would like to thank Professor Sadahiro Maeda for his valuable suggestions and encouragement during the preparation of this doctoral paper.

1. INTRODUCTION

We denote by $\widetilde{M}_n(c)$ a complex $n(\geqq 2)$-dimensional complete and simply connected Kähler manifold of constant holomorphic sectional curvature c, namely a complex space form of constant holomorphic sectional curvature c. It is well-known that $\widetilde{M}_n(c)$ is holomorphically isometric to either an n-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c, a complex Euclidean space $\mathbb{C}^n$ or an n-dimensional complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature c according as c is positive, zero or negative.

In the theory of real hypersurfaces in a nonflat complex space form $\widetilde{M}_n(c)$, R. Takagi, M. Kimura, J. Berndt, S. Maeda, H. Tamaru have made significant contributions (see [18, 19, 11, 5, 6, 1, 7, 12]). M. Kimura ([11]) classified Hopf hypersurfaces M all of whose principal curvatures are constant in $\mathbb{C}P^n(c)$ ($n \geqq 2$). In the case of $\mathbb{C}H^n(c)$ ($n \geqq 2$), J. Berndt classified Hopf hypersurface M all of whose principal curvatures are constant. Moreover, J. Berndt and H. Tamaru ([6]) classified all homogeneous real hypersurfaces in $\mathbb{C}H^n(c)$ ($n \geqq 2$). They showed that there exist many homogeneous real hypersurfaces which are not Hopf hypersurfaces as well as many homogeneous real hypersurfaces which are Hopf hypersurfaces. S. Maeda, T. Adachi, M. Kimura, and B.Y. Chen ([1, 7]) characterized all homogeneous Hopf hypersurfaces M by studying the holomorphic distribution $T^0M := \{X \in TM | X \perp \xi\}$ of M and the extrinsic shape of some geodesics on M in a nonflat complex space form, where TM is the tangent bundle over M, and ξ is called the characteristic vector field on M.

In Section 2, we prepare basic terminologies on real hyperspaces and some fundamental properties of Hopf hypersurfaces in a nonflat space form $\widetilde{M}_n(c)$.

In Section 3, we review M. Kimura's and J. Berndt's classification Theorems of Hopf hypersurfaces all of whose principal curvatures are constant in a nonflat complex space form.

In Section 4, we consider the extrinsic shape of geodesics on a hypersurface M^n in an ambient Riemannian mainifold $\widetilde{M}^{n+1}$ through an isometric immersion. In general, it is meaningful to investigate the extrinsic shape of geodesics on a hypersurface M^n in the ambient space $\widetilde{M}^{n+1}$. We here call $\iota \circ \gamma$ the extrinsic shape of a curve γ on M^n in $\widetilde{M}^{n+1}$, where ι denotes an isometric immersion of M^n into $\widetilde{M}^{n+1}$. In Section 4, from this point of view, we review several results proved by S. Maeda, T. Adachi and Y.H. Kim (see [14]).

The main result of the first part is stated in Section 5. We characterize the homogeneous real hypersurface M of type (B) with two distinct principal curvatures in $\mathbb{C}H^n(c)$ in terms of the derivative of its shape operator A, the exterior differentiation $d\eta$ of its contact form η and the extrinsic shape of some geodesics on M. We here note that there does not exist a real hypersurface M with parallel shape operator A, and also does not exist a real hypersurface M with closed contact form η in $\widetilde{M}_n(c)$. We here review homogeneous real hypersurfaces M of type (B) in $\mathbb{C}H^n(c)$. This real hypersurface M is a tube of constant radius $r \in (0, \infty)$ around a totally real totally geodesic $\mathbb{R}H^n(c/4)$ in the ambient space $\mathbb{C}H^n(c)$. When $r \neq (1/\sqrt{|c|}\,)\log(2+\sqrt{3}\,)$, M has three distinct constant principal curvatures $\lambda_1 = (\sqrt{|c|}\,/2)\coth(\sqrt{|c|}\,r/2)$, $\lambda_2 = (\sqrt{|c|}\,/2)\tanh(\sqrt{|c|}\,r/2)$ and $\delta = \sqrt{|c|}\,\tanh(\sqrt{|c|}\,r)$. In the case of $r = (1/\sqrt{|c|}\,)\log(2+\sqrt{3}\,)$, M has two distinct constant principal curvatures $\lambda_1 = \delta = \sqrt{3|c|}\,/2$ and $\lambda_2 = \sqrt{|c|}\,/(2\sqrt{3}\,)$.

We next explain the second part. The geometry of real hypersurfaces of $\widetilde{M}_n(c)$, $c \neq 0$ is a bit complicated. For examples, the following three properties are known.

(i) There exist no real hypersurfaces M with parallel shape operator A.
(ii) There exist no real hypersurfaces M with parallel Ricci tensor S.
(iii) There exist no real hypersurfaces M with parallel structure tensor ϕ (As for (i), (ii), see [17] and for (iii), see Proposition 1.)

As we mention in Section 2, by the complex structure J of $\widetilde{M}_n(c)$ the structure tensor ϕ is defined on real hypersurfaces M of $\widetilde{M}_n(c)$. We here define a symmetric tensor $\psi = \phi A - A\phi$ on M. Using the parallelism of this tensor ψ, we shall characterize homogeneous real hypersurfaces of type (A), which are the simplest examples of Hopf hypersurfaces with constant principal curvatures in a nonflat complex space form. There are many conditions which characterize homogeneous real hypersurfaces of type (A) in a nonflat complex space form. Among them, the condition that $\psi = 0$ on M is well known. (see Propsition 3).
In the case of $c > 0$, T. Hamada ([9]) proved that a real hypersurface M is of type (A) if and only if the tensor ψ on M is parallel, which is a generalization of the condition $\psi = 0$. In Theorem 4 we show that this result holds also in the case of $c < 0$. Using the discussion in the proof of Theorem 4, we can prove Theorem 5. In Theorem 5 we give a necessary and sufficient condition for a Kähler manifold to be a complex space form. Theorems 4, 5 are the main results of the second part and they are stated in Section 6. In Sections 7 and 8 we give the proofs of Theorem 4 and Theorem 5, respectively.

2. BASIC FACTS AND FUNDAMENTAL PROPERTIES

Let M^{2n-1} be a real hypersurface of a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$.

Before dealing with real hypersurfaces of $\widetilde{M}_n(c)$, we review basic facts about $\mathbb{C}P^n(c)$ and $\mathbb{C}H^n(c)$.

We first explain the geometric structure of $\mathbb{C}P^n(c)$. Without loss of generality we can set $c = 4$. We consider the Hermitian form $\langle\langle\ ,\ \rangle\rangle$ on the complex vector space $\mathbb{C}^{n+1}$ given by

$$\langle\langle z, w\rangle\rangle = \sum_{j=0}^{n} z_j \overline{w_j},$$

where $z = (z_0, \ldots, z_n),\ w = (w_0, \ldots, w_n)$. We define a Riemmanian metric $\langle\ ,\ \rangle$ on $\mathbb{C}^{n+1}$ by

$$\langle z, w\rangle = \mathrm{Re}\langle\langle z, w\rangle\rangle.$$

We consider the hypersurface S^{2n+1} of $\mathbb{C}^{n+1}$ defined by

$$S^{2n+1} = \{z \in \mathbb{C}^{n+1} | \langle\langle z, z\rangle\rangle = 1\}$$

endowed with the metric induced from $\langle\ ,\ \rangle$. S^{2n+1} is a principal S^1-bundle over $\mathbb{C}P^n$ with projection mapping $\pi : S^{2n+1} \to \mathbb{C}P^n(4)$. For $z \in S^{2n+1}$, the tangent space $T_z S^{2n+1} = \{w \in \mathbb{C}^{n+1} | \langle w, z\rangle = 0\}$ and the tangent space $T_{\pi(z)}\mathbb{C}P^n$ can be identified with the subspace of $\mathbb{C}^{n+1}$

$$T_{\pi(z)}\mathbb{C}P^n = \{w \in \mathbb{C}^{n+1} | \langle\langle w, z\rangle\rangle = 0\}.$$

We next explain the geometric structure of $\mathbb{C}H^n(c)$ in detail. We set $c = -4$. We consider the Hermitian form $\langle\langle\ ,\ \rangle\rangle$ on the complex vector space $\mathbb{C}^{n+1}$ given by

$$\langle\langle z, w\rangle\rangle = -z_0\overline{w_0} + \sum_{j=1}^{n} z_j \overline{w_j},$$

where $z = (z_0, \ldots, z_n),\ w = (w_0, \ldots, w_n)$. We define an indefinite inner product $\langle\ ,\ \rangle$ on $\mathbb{C}^{n+1}$ by

$$\langle z, w\rangle = \mathrm{Re}\langle\langle z, w\rangle\rangle.$$

We consider the hypersurface $H_1^{2n+1}(-1)$ of $\mathbb{C}^{n+1}$ defined by

$$H_1^{2n+1}(-1) = \{z \in \mathbb{C}^{n+1} | \langle\langle z, z\rangle\rangle = -1\}$$

endowed with the metric induced from $\langle\ ,\ \rangle$. For $z \in H_1^{2n+1}(-1)$, the tangent space $T_z H_1^{2n+1}(-1) = \{w \in \mathbb{C}^{n+1} | \langle w, z\rangle = 0\}$. At $z \in H_1^{2n+1}$, for the normal vector $N_z = z$, we have

$$\langle JN_z, N_z\rangle = \mathrm{Re}\langle\langle JN_z, N_z\rangle\rangle = -\langle JN_z, N_z\rangle.$$

Therefore, $\langle JN_z, N_z\rangle = 0$. This means $JN_z \in T_z H_1^{2n+1}(-1)$. Moreover,

$$\langle JN_z, JN_z\rangle = \mathrm{Re}\langle\langle JN_z, JN_z\rangle\rangle = \mathrm{Re}\langle\langle N_z, N_z\rangle\rangle = -1.$$

Therefore, $H_1^{2n+1}(-1)$ is not a Riemmanian manifold.
So, the tangent space $T_{\pi(z)}\mathbb{C}H^n$ can be identified with the subspace of $\mathbb{C}^{n+1}$

$$T_{\pi(z)} = \{w \in \mathbb{C}^{n+1} | \langle\langle w, z\rangle\rangle = 0\}.$$

The complex structure J of $\mathbb{C}H^n$ is defined by $JX = (\pi_*)_z(\sqrt{-1}\, X')$, where $X \in T_{\pi(z)}\mathbb{C}H^n$ and X' is a horizontal lift for X. It is well-known that the Bergman metric g of constant holomorphic sectional curvature -4 is given by $g(X, Y) = \langle X', Y'\rangle$, where $X, Y \in T_z(\mathbb{C}H^n(-4))$ and X', Y' are respectively their horizontal lifts.

Now, we shall review the fundamental properties of real hypersurfaces of $\widetilde{M}_n(c)$. A real hypersurface M^{2n-1} has locally a unit normal vector field $\mathcal{N}$. The Riemannian connections $\widetilde{\nabla}$ of $\widetilde{M}_n(c)$ and ∇ of M are related by the following formulas of Gauss and Weingarten:

$$\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}, \tag{2.1}$$

$$\widetilde{\nabla}_X \mathcal{N} = -AX \tag{2.2}$$

for arbitrary vector fields X and Y on M, where g is the Riemannian metric induced from the standard metric of the ambient space $\widetilde{M}_n(c)$ and A is the shape operator of M in $\widetilde{M}_n(c)$. Due to the property $g(AX, Y) = g(X, AY)$, A is symmetric. So its eigenvalues are real numbers. They are called *principal curvatures* of M in $\widetilde{M}_n(c)$. Eigenvectors of the shape operator A are called *principal curvature vectors* of M in $\widetilde{M}_n(c)$. $V_\lambda = \{v \in TM | Av = \lambda v\}$ is called the principal distribution associated to the principal curvature λ.

It is well-known that M has an *almost contact metric structure* (ϕ, ξ, η, g) induced from the Kähler structure J of the ambient space $\widetilde{M}_n(c)$. This quadruple is defined by

$$g(\phi X, Y) = g(JX, Y), \quad \xi = -J\mathcal{N} \quad \text{and} \quad \eta(X) = g(\xi, X) = g(JX, \mathcal{N}).$$

The vector field ξ is called *the characteristic vector field* on M. This quadruple satisfies

$$\phi^2 = -I + \eta \otimes \xi,\ g(\xi, \xi) = 1 \quad \text{and} \quad \phi\xi = 0 \tag{2.3}$$

It follows from (2.1), (2.2), $\widetilde{\nabla} J = 0$ and $JX = \phi X + \eta(X)\mathcal{N}$ that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{2.4}$$

$$\nabla_X \xi = \phi AX. \tag{2.5}$$

Indeed, for (2.5), we get

$$\begin{aligned}
\nabla_X \xi &= -\nabla_X (J\mathcal{N}) = -\widetilde{\nabla}_X (J\mathcal{N}) + g(AX, J\mathcal{N})\mathcal{N} \\
&= -J\widetilde{\nabla}_X \mathcal{N} + g(AX, J\mathcal{N})\mathcal{N} = JAX - g(JAX, \mathcal{N})\mathcal{N} = \phi AX.
\end{aligned}$$

And for (2.4), we see

$$\begin{aligned}
(\nabla_X \phi)Y &= \nabla_X(\phi Y) - \phi\nabla_X Y = \nabla_X(JY - \eta(Y)\mathcal{N}) - \phi\nabla_X Y \\
&= \widetilde{\nabla}_X(JY - \eta(Y)\mathcal{N}) - g(A\phi Y, X)\mathcal{N} - \phi\nabla_X Y \\
&= J(\nabla_X Y + g(AX, Y)\mathcal{N}) - X(\eta(Y))\mathcal{N} + \eta(Y)AX \\
&\qquad - g(A\phi Y, X)\mathcal{N} - \phi\nabla_X Y \\
&= \phi\nabla_X Y + g(\nabla_X Y, \xi)\mathcal{N} - g(AX, Y)\xi - g(\nabla_X Y, \xi)\mathcal{N} \\
&\quad - g(Y, \phi AX)\mathcal{N} + \eta(Y)AX - g(A\phi Y, X)\mathcal{N} - \phi\nabla_X Y \\
&= \eta(Y)AX - g(AX, Y)\xi.
\end{aligned}$$

For the curvature tensors R of the hypersurface M and $\widetilde{R}$ of the ambient space $\widetilde{M}_n(c)$, we have the equation of Gauss

$$g(R(X,Y)Z,W) = g(\widetilde{R}(X,Y)Z,W) + g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W). \tag{2.6}$$

Since the curvature tensor $\widetilde{R}$ is written as

$$\widetilde{R}(X,Y)Z = \frac{c}{4}\{g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX - g(JX,Z)JY + 2g(X,JY)JZ\}, \tag{2.7}$$

from (2.6) the equation of Gauss is reduced to

$$\begin{aligned} g(R(X,Y)Z,W) = \frac{c}{4}\{&g(Y,Z)g(X,W) - g(X,Z)g(Y,W) \\ &+ g(JY,Z)g(JX,W) - g(JX,Z)g(JY,W) - 2g(\phi X,Y)g(\phi Z,W)\} \\ &+ g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W)\}. \end{aligned} \tag{2.8}$$

It follows from

$$g(\widetilde{R}(X,Y)Z,\mathcal{N}) = g((\nabla_X A)Y - (\nabla_Y A)X, Z)$$

and (2.7) that

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X,Y)\xi\}. \tag{2.9}$$

We call (2.9) the Codazzi equation for a real hypersurface M in $\widetilde{M}_n(c)$. We usually call M *a Hopf hypersurface* if the characteristic vector field ξ of M is a principal curvature vector at each point of M. In the following, for a Hopf hypersurface M we set $A\xi = \delta\xi$ on M. Here, we review the following lemma which is a useful tool in the theory of Hopf hypersurfaces in a nonflat complex space form (cf. [15, 17]).

Lemma 1. *For a Hopf hypersurface M^{2n-1} $(n \geqq 2)$ with principal curvature δ corresponding to the characteristic vector field ξ in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$, we have the following.*

(1) *If X is a tangent vector of M perpendicular to ξ with $AX = \lambda X$, then $(2\lambda - \delta)A\phi X = (\delta\lambda + (c/2))\phi X$.*
(2) *δ is constant locally on M.*

Proof. We adopt the discussion in the proof of this lemma in [17].

(1) It follows from (2.5) and $A\xi = \delta\xi$ that

$$(\nabla_X A)\xi = \nabla_X(A\xi) - A\nabla_X\xi = (X\delta)\xi + (\delta I - A)\phi AX. \tag{2.10}$$

This, together with (2.9), shows

$$X\delta = g((\nabla_X A)\xi,\xi) = g((\nabla_\xi A)X,\xi). \tag{2.11}$$

So, from $g((\nabla_\xi A)X,\xi) = g((\nabla_\xi A)\xi,X) = (\xi\delta)\eta(X)$ we see that $X\delta = 0$ for all vectors X perpendicular to ξ, so that grad $\delta = (\xi\delta)\xi$. Now, using (2.10) and (2.11), we have

$$\begin{aligned} g((\nabla_X A)Y,\xi) &= g((\nabla_X A)\xi,Y) \\ &= (\xi\delta)\eta(X)\eta(Y) + g((\delta I - A)\phi AX,Y). \end{aligned} \tag{2.12}$$

Exchanging X and Y in (2.12) and subtracting these equations, we compute

$$g((\nabla_X A)Y,\xi) - g((\nabla_Y A)X,\xi) = g((\delta I - A)\phi AX, Y) - g((\delta I - A)\phi AY, X).$$

This, combined with (2.9), implies

$$\begin{aligned}(c/2)g(X,\phi Y) &= g((\delta I - A)\phi AX, Y) - g((\delta I - A)\phi AY, X)\\ &= -g(X, A\phi(\delta I - A)Y) - g(X, (\delta I - A)\phi AY)\end{aligned}$$

for all $X, Y \in TM$. Thus we can see that

$$A\phi A - (\delta/2)(A\phi + \phi A) - (c/4)\phi = 0. \tag{2.13}$$

Statement (1) is an immediate consequence of (2.13).

(2) Let $\beta = \xi\delta$. Then grad $\delta = \beta\xi$ (see the proof of Statement (1)). We have

$$\begin{aligned}&g(\nabla_X(\text{grad }\delta), Y) - g(\nabla_Y(\text{grad }\delta), X)\\ &\quad= X(g(\text{grad }\delta, Y)) - g(\text{grad }\delta, \nabla_X Y) - Y(g(\text{grad }\delta, X))\\ &\qquad\quad + g(\text{grad }\delta, \nabla_Y X)\\ &\quad= XY\delta - YX\delta - g(\text{grad }\delta, \nabla_X Y - \nabla_Y X)\\ &\quad= ([X,Y] - (\nabla_X Y - \nabla_Y X))\delta = 0.\end{aligned}$$

This, together $\beta = \xi\delta$, yields

$$\begin{aligned}0 &= g(\nabla_X(\beta\xi), Y) - g(\nabla_Y(\beta\xi), X)\\ &= X\beta\eta(Y) + \beta g(\phi AX, Y) - Y\beta\eta(X) - \beta g(\phi AY, X)\\ &= (X\beta)\eta(Y) - (Y\beta)\eta(X) + \beta g((\phi A + A\phi)X, Y).\end{aligned} \tag{2.14}$$

Setting $Y = \xi$ in (2.14), we get $0 = X\beta - (\xi\beta)\eta(X)$, where we have used $A\xi = \delta\xi$ and $\phi\xi = 0$. Thus we see that $X\beta = (\xi\beta)\eta(X)$ for all vectors X. This, combined with (2.14), shows

$$(\xi\delta)(\phi A + A\phi) = 0. \tag{2.15}$$

Note that Equation (2.15) is a key in the proof of Statement (2). In the following, we suppose that $\xi\delta \neq 0$ at some point. Then it follows from (2.15) that $\phi A + A\phi = 0$ in a sufficiently small neighborhood of this point. So, from (2.13) we know that $\phi A^2 + (c/4)\phi = 0$. Now, applying this equation to a principal curvature vector X orthogonal to ξ, we get

$$0 = \phi(A^2 + (c/4)I)X = (\lambda^2 + (c/4))\phi X,$$

where λ is the principal curvature for X. Hence $\lambda^2 + (c/4) = 0$. Then we obtain a contradiction in the case of $c > 0$. Thus we find that grad $\delta = 0$, namely δ is constant locally on M when c is positive.

Therefore the rest of the proof is to verify that $\xi\delta = 0$ also holds on M in the case of $c < 0$. Suppose that $\phi A + A\phi = 0$. So we can use $\phi(A^2 + (c/4)I) = 0$. Hence

$$\begin{aligned}0 &= (\nabla_X(\phi(A^2 + (c/4)I)))Y\\ &= (\nabla_X\phi)(A^2 + (c/4)I)Y + \phi(\nabla_X A)AY + \phi A(\nabla_X A)Y.\end{aligned} \tag{2.16}$$

Hence, from (2.4), Equation (2.16) becomes

$$0 = (\delta^2 + (c/4))\eta(Y)AX - g((A^3 + (c/4)A)X, Y)\xi + \phi(\nabla_X A)AY \\ + \phi A(\nabla_X A)Y.$$

Applying ϕ to this equality, we get

$$\phi((\delta^2 + (c/4))\eta(Y)AX) + \phi^2((\nabla_X A)AY) + \phi^2(A(\nabla_X A)Y) = 0. \tag{2.17}$$

The second term of (2.17) is rewritten as

$$\phi^2((\nabla_X A)AY) = -(\nabla_X A)AY + g((\nabla_X A)AY, \xi)\xi.$$

It follows from (2.5), $g((\nabla_X A)Y, Z) = g(Y, (\nabla_X A)Z)$ and $\phi A^2 = -(c/4)\phi$ that

$$\begin{aligned} g((\nabla_X A)AY, \xi) &= g(AY, (X\delta)\xi + (\delta I - A)\phi AX) \\ &= \delta\beta\eta(X)\eta(Y) + \delta g(A\phi AX, Y) - g(A^2\phi AX, Y) \\ &= \delta\beta\eta(X)\eta(Y) + (c/4)\delta g(\phi X, Y) + (c/4)g(\phi AX, Y). \end{aligned}$$

Again using $\phi^2 X = -X + g(X, \xi)\xi$, we can rewrite the third term of (2.17) as

$$\phi^2(A(\nabla_X A)Y) = -A(\nabla_X A)Y + g(A(\nabla_X A)Y, \xi)\xi,$$

and by a direct computation we see that

$$\begin{aligned} g(A(\nabla_X A)Y, \xi) &= \delta g((\nabla_X A)Y, \xi) \\ &= \delta\beta\eta(X)\eta(Y) + \delta^2 g(\phi AX, Y) - (c\delta/4)g(\phi X, Y). \end{aligned}$$

Then by all of the above computation we know that

$$\begin{aligned} (\nabla_X A)AY + A(\nabla_X A)Y = {} & 2\delta\beta\eta(X)\eta(Y)\xi \\ & + (\delta^2 + (c/4))(g(\phi AX, Y)\xi + \eta(Y)\phi AX). \end{aligned} \tag{2.18}$$

Here, exchanging X and Y in (2.18) and subtracting these equations, from (2.9) and the equality $\phi A + A\phi = 0$ we know that

$$\begin{aligned} (\nabla_X A)AY - (\nabla_Y A)AX = (c\delta/2)g(\phi X, Y)\xi + \delta^2(\eta(Y)\phi AX \\ - \eta(X)\phi AY). \end{aligned} \tag{2.19}$$

Taking the inner product of $(\nabla_X A)AY$ and Z, from the symmetry of A, $\phi A + A\phi = 0$ and (2.9) we have

$$\begin{aligned} & g((\nabla_X A)AY, Z) = g(AY, (\nabla_X A)Z) \\ & \quad = g(AY, (\nabla_Z A)X) + (c/4)(\eta(X)g(A\phi Y, Z) - \eta(Z)g(A\phi X, Y) + 2\delta\eta(Y)g(X, \phi Z)). \end{aligned}$$

Exchanging X and Y in this equation and subtracting the two equations, we obtain

$$\begin{aligned} g((\nabla_X A)AY, Z) - g((\nabla_Y A)AX, Z) = g(AY, (\nabla_Z A)X) \\ - g(AX, (\nabla_Z A)Y) + (c/4)(\eta(X)g(A\phi Y, Z) - \eta(Y)g(A\phi X, Z)) \\ + (c\delta/2)(\eta(Y)g(X, \phi Z) - \eta(X)g(Y, \phi Z)). \end{aligned}$$

Then the coefficient of X on the right hand side of this equation is

$$\begin{aligned} (\nabla_Z A)AY - A(\nabla_Z A)Y + (c/4)(g(A\phi Y, Z)\xi - \eta(Y)A\phi Z) \\ + (c\delta/2)(\eta(Y)\phi Z - g(Y, \phi Z)\xi). \end{aligned} \tag{2.20}$$

On the other hand, taking the inner product of (2.19) and Z, we find that the coefficient of X on the right hand side is

$$-(c\delta/2)\eta(Z)\phi Y + \delta^2(\eta(Y)\phi AZ - g(\phi AY, Z)\xi).$$

This, together with (2.20), yields

$$\begin{aligned}(\nabla_Z A)AY - A(\nabla_Z A)Y &= (\delta^2 - (c/4))\eta(Y)\phi AZ \\ &- (c\delta/2)(\eta(Y)\phi Z + \eta(Z)\phi Y + g(\phi Y, Z)\xi) - (\delta^2 - (c/4))g(\phi AY, Z)\xi.\end{aligned}$$

Replacing Z with X in this equation, we have

$$\begin{aligned}(\nabla_X A)AY - &A(\nabla_X A)Y \\ &= (\delta^2 - (c/4))(\eta(Y)\phi AX - g(\phi AX, Y)\xi) \\ &\quad - (c\delta/2)(\eta(Y)\phi X + \eta(X)\phi Y + g(\phi Y, X)\xi).\end{aligned} \tag{2.21}$$

It follows from (2.18) and (2.21) that

$$\begin{aligned}(\nabla_X A)AY = \beta\delta\eta(X)\eta(Y)\xi + (c/4)g(\phi AX, Y)\xi + \delta^2\eta(Y)\phi AX \\ - (c\delta/4)(\eta(Y)\phi X + \eta(X)\phi Y + g(\phi Y, X)\xi).\end{aligned} \tag{2.22}$$

Also recall that $A\phi A = (c/4)\phi$. Replacing Y by AY in (2.22), we get

$$\begin{aligned}(\nabla_X A)A^2Y &= \beta\delta^2\eta(X)\eta(Y)\xi + (c^2/16)g(\phi X, Y)\xi \\ &+ \delta^3\eta(Y)\phi AX - (c\delta^2/4)\eta(Y)\phi X - (c\delta/4)\eta(X)\phi AY \\ &- (c\delta/4)g(\phi AY, X)\xi.\end{aligned} \tag{2.23}$$

We note that $(A^2 + (c/4)I)Y = (\delta^2 + (c/4))\eta(Y)\xi$, since $\phi(A^2 + (c/4)I) = 0$. This shows that $A^2Y = (-c/4)Y + (\delta^2 + (c/4))\eta(Y)\xi$. So we can compute directly the following equalities.

$$\begin{aligned}(\nabla_X A)A^2Y &= (-c/4)(\nabla_X A)Y + (\delta^2 + (c/4))\eta(Y)(\nabla_X A)\xi \\ &= (-c/4)(\nabla_X A)Y + (\delta^2 + (c/4))\beta\eta(Y)\eta(X)\xi \\ &\quad + (\delta^2 + (c/4))\eta(Y)\delta\phi AX - (\delta^2 + (c/4))(c/4)\eta(Y)\phi X \\ &= (-c/4)(\nabla_X A)Y + \beta\delta^2\eta(X)\eta(Y)\xi + (c\beta/4)\eta(X)\eta(Y)\xi \\ &\quad + \delta^3\eta(Y)\phi AX + (c\delta/4)\eta(Y)\phi AX - (c\delta^2/4)\eta(Y)\phi X \\ &\quad - (c^2/16)\eta(Y)\phi X.\end{aligned}$$

This, combined with (2.23), shows

$$\begin{aligned}(\nabla_X A)Y = \beta\eta(X)\eta(Y)\xi + \delta(\eta(X)\phi AY + \eta(Y)\phi AX \\ + g(\phi AX, Y)\xi) + (c/4)(g(\phi Y, X)\xi - \eta(Y)\phi X).\end{aligned} \tag{2.24}$$

We shall compute $(R(X, \phi X) \cdot A)Z$ for each X orthogonal to ξ by using (2.24), which is defined by

$$(R(X, \phi X) \cdot A)Z = R(X, \phi X)(AZ) - A(R(X, \phi X)Z),$$

where R is the curvature tensor of our real hypersurface M. By a direct calculation we find

$$
\begin{aligned}
(2.25)\quad &\nabla_X((\nabla_{\phi X}A)Z)\\
&= \nabla_X(\delta(\eta(Z)\phi A\phi X + g(\phi A\phi X, Z)\xi) + (c/4)(g(\phi Z, \phi X)\xi - \eta(Z)\phi^2 X))\\
&= \delta(g(\nabla_X Z, \xi)AX + g(Z, \nabla_X\xi)AX + \eta(Z)\nabla_X(AX)\\
&\qquad + g(\nabla_X(AX), Z)\xi + g(AX, \nabla_X Z)\xi + g(AX, Z)\nabla_X\xi)\\
&\quad + (c/4)(g(\nabla_X X, Z)\xi + g(X, \nabla_X Z)\xi + g(X, Z)\nabla_X\xi\\
&\qquad + g(\nabla_X\xi, Z)X + g(\xi, \nabla_X Z)X + \eta(Z)\nabla_X X),
\end{aligned}
$$

where we have used $X\delta = \eta(X)\beta$. Here, from (2.24) we see that

$$\nabla_X(AX) = A(\nabla_X X) + (\nabla_X A)X = A(\nabla_X X) + \delta g(\phi AX, X)\xi.$$

Then we rewrite (2.25) as

$$
\begin{aligned}
(2.26)\quad &\nabla_X((\nabla_{\phi X}A)Z)\\
&= \delta(g(\nabla_X Z, \xi)AX + g(Z, \phi AX)AX + \eta(Z)A(\nabla_X X)\\
&\quad + \delta g(\phi AX, X)\eta(Z)\xi + g(A(\nabla_X X), Z)\xi + \delta g(\phi AX, X)\eta(Z)\xi\\
&\qquad + g(AX, \nabla_X Z)\xi + g(AX, Z)\phi AX)\\
&\quad + (c/4)(g(\nabla_X X, Z)\xi + g(X, \nabla_X Z)\xi + g(X, Z)\phi AX\\
&\qquad + g(\phi AX, Z)X + g(\xi, \nabla_X Z)X + \eta(Z)\nabla_X X).
\end{aligned}
$$

Moreover, we have similarly

$$
\begin{aligned}
(2.27)\quad (\nabla_{\phi X}A)(\nabla_X Z) &= \delta(g(\nabla_X Z, \xi)AX + g(AX, \nabla_X Z)\xi)\\
&\quad + (c/4)(g(\nabla_X Z, X)\xi + g(\xi, \nabla_X Z)X)
\end{aligned}
$$

and

$$
\begin{aligned}
(2.28)\quad &(\nabla_{\nabla_X\phi X}A)Z\\
&= -\beta g(AX, X)\eta(Z)\xi + \delta(\eta(Z)A\nabla_X X - g(AX, X)\phi AZ\\
&\quad - \eta(Z)g(\nabla_X X, \xi)\delta\xi + g(A\nabla_X X, Z)\xi - \delta g(\nabla_X X, \xi)\eta(Z)\xi)\\
&\quad + (c/4)(g(Z, \nabla_X X)\xi - \eta(Z)g(\nabla_X X, \xi)\xi + \eta(Z)\nabla_X X\\
&\qquad - \eta(Z)g(\nabla_X X, \xi)\xi).
\end{aligned}
$$

We now define

$$
\begin{aligned}
N(X, Z) &= (\nabla_X\nabla_{\phi X}A - \nabla_{\nabla_X\phi X}A)Z\\
&= \nabla_X((\nabla_{\phi X}A)Z) - (\nabla_{\phi X}A)(\nabla_X Z) - (\nabla_{\nabla_X\phi X}A)Z.
\end{aligned}
$$

This, together with (2.26), (2.27) and (2.28), implies

$$
\begin{aligned}
(2.29)\quad &N(X, Z) = \beta g(AX, X)\eta(Z)\xi\\
&\quad + \delta(g(Z, \phi AX)AX + g(AX, Z)\phi AX + g(AX, X)\phi AZ)\\
&\quad + (c/4)(g(X, Z)\phi AX + g(\phi AX, Z)X - 2\eta(Z)g(\phi AX, X)\xi).
\end{aligned}
$$

Since X is perpendicular to ξ, by the definition of N we get

$$\begin{aligned} N(\phi X, Z) &= (-\nabla_{\phi X}\nabla_X A + \nabla_{\nabla_{\phi X}X} A)Z \\ &= (R(X, \phi X)\cdot A)Z - (\nabla_X \nabla_{\phi X} A - \nabla_{\nabla_X \phi X} A)Z, \end{aligned}$$

so that

$$(R(X, \phi X)\cdot A)Z = N(X, Z) + N(\phi X, Z).$$

On the other hand, from (2.29) we know that

$$\begin{aligned} N(\phi X, Z) = {} & -\beta g(AX, X)\eta(Z)\xi \\ & + \delta(g(Z, AX)A\phi X + g(A\phi X, Z)AX - g(AX, X)\phi AZ) \\ & + (c/4)(g(\phi X, Z)AX + g(AX, Z)\phi X - 2\eta(Z)g(X, A\phi X)\xi). \end{aligned}$$

Hence

$$\begin{aligned} (R(X, \phi X)\cdot A)Z = (c/4)(g(X, Z)\phi AX + g(X, \phi AZ)X - g(X, \phi Z)AX \\ + g(X, AZ)\phi X). \end{aligned}$$

Now let $\{e_i\}$ be an orthonormal basis of $\xi^{\perp}$. Then we have

$$\begin{aligned} \sum (R(e_i, \phi e_i)\cdot A)Z &= (c/4)(\phi AZ + \phi AZ - A\phi Z + \phi AZ) \\ &= c\phi AZ. \end{aligned} \tag{2.30}$$

We consider the following for any (1, 1) tensor T

$$(TX \wedge T\phi X)AZ - A(TX \wedge T\phi X)Z,$$

where $(X \wedge Y)Z = g(Y, Z)X - g(X, Z)Y$. In general, we find

$$\begin{aligned} & (TX \wedge T\phi X)AZ - A(TX \wedge T\phi X)Z \\ & \quad = g(T\phi X, AZ)TX - g(TX, AZ)T\phi X - g(T\phi X, Z)ATX \\ & \qquad + g(TX, Z)AT\phi X. \end{aligned} \tag{2.31}$$

Summing (2.31) over $X = e_i$, we can see that the right hand side becomes

$$\begin{aligned} -T(\phi T^* AZ) - T\phi T^* AZ + AT\phi T^* Z + AT\phi T^* Z \\ = -2T\phi T^* AZ + 2AT\phi T^* Z, \end{aligned} \tag{2.32}$$

where T^* is the transpose of T. In the case of $T = I$, (2.32) becomes

$$2(A\phi - \phi A)Z = -4\phi AZ.$$

When $T = A$, (2.32) is

$$-2A\phi A^2 Z + 2A^2\phi AZ = -2(A\phi A)AZ + 2A(A\phi A)Z = -c\phi AZ.$$

Here we have used $A\phi A = (c/4)\phi$. It follows from (2.8) that

$$\begin{aligned} R(e_i, \phi e_i) &= Ae_i \wedge A\phi e_i + (c/4)(e_i \wedge \phi e_i + \phi e_i \wedge \phi^2 e_i + 2g(e_i, \phi^2 e_i)\phi) \\ &= Ae_i \wedge A\phi e_i + (c/2)(e_i \wedge \phi e_i) - (c/2)\phi. \end{aligned}$$

Since $(R(e_i, \phi e_i)\cdot A)Z = R(e_i, \phi e_i)(AZ) - AR(e_i, \phi e_i)Z$, by the summation of the last term in $(R(e_i, \phi e_i)\cdot A)Z$ gives $-c(2n-2)\phi AZ$. Using this and (2.32) with $T = I$ and $T = A$, we see that

$$\sum (R(e_i, \phi e_i)\cdot A)Z = -c(2n+1)\phi AZ. \tag{2.33}$$

For all tangent vectors Z, from (2.30) and (2.33) we find that

$$2c(n+1)\phi AZ = 0,$$

so that $\phi A = 0$. This implies that $AX = \eta(AX)\xi$ for all $X \in TM$. Hence, from (2.5) we know that

$$(\nabla_X A)Y = \nabla_X(AY) - A\nabla_X Y \in \mathrm{span}\{\xi\},$$

which, together with (2.9), yields

$$(c/4)(\eta(X)\phi Y - \eta(Y)\phi X) \in \mathrm{span}\{\xi\}.$$

Putting $Y = \xi$ in this equation, we can see that $(-c/4)\phi X \in \mathrm{span}\{\xi\}$ for all $X \in TM$. Thus we obtain a contradiction. Therefore, in the case of $c < 0$ we conclude that δ is constant locally on M. $\square$

The discussion in the proof of Lemma 1 gives the following fundamental fact in the study of real hypersurfaces in $\widetilde{M}_n(c), n \geqq 2$.

Lemma 2. *There exist no real hypersurfaces with $\phi A + A\phi = 0$ in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$.*

By virtue of Lemma 2, we obtain the following fundamental property of all Hopf hypersurfaces in a nonflat complex space form.

Lemma 3 ([7])**.** *For every Hopf hypersurface M in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$, the holomorphic distribution $T^0M = \{X \in TM | \eta(X) = 0\}$ on M is not integrable.*

Proof. Suppose that there exist a Hopf hypersurface M with the integrable holomorphic distribution T^0M in $\widetilde{M}_n(c)$. Note that T^0M is integrable if and only if

$$[X, Y] = \nabla_X Y - \nabla_Y X \in T^0M \quad \text{for all } X, Y \in T^0M.$$

Hence, for all $X, Y \in T^0M$ from (2.5) we have

$$\begin{aligned} 0 &= g(\nabla_X Y - \nabla_Y X, \xi) = -g(Y, \nabla_X \xi) + g(X, \nabla_Y \xi) \\ &= -g(Y, \phi AX) + g(X, \phi AY) = g(X, (\phi A + A\phi)Y), \end{aligned}$$

which implies that T^0M is integrable if and only if

$$g((\phi A + A\phi)X, Y) = 0 \quad \text{for all } X, Y \in T^0M. \tag{2.34}$$

This, combined with the assumption that ξ is principal, shows that $\phi A + A\phi = 0$ holds on our real hypersurface M. This is a contradiction. $\square$

In this context, it is natural to consider a problem that does there exist a Hopf hypersurface M in $\widetilde{M}_n(c)$ satisfying that T^0M is decomposed as the direct sum of integrable distributions? The following lemma gives a characterization of all real hypersurfaces of type (B) from this viewpoint ([7, 12]).

Lemma 4. *A connected real hypersurface M^{2n-1} (with Riemannian connection ∇) of $\mathbb{C}H^n(c), n \geqq 2$ is of type (B) if and only if M satisfies the following two conditions*:

1. *The holomorphic distribution T^0M of M is decomposed as the direct sum of restricted principal distributions $V^0_{\lambda_i} = \{X \in T^0M | AX = \lambda_i X\}$;*

(2) *Every* $V^0_{\lambda_i}$ *in Condition* (1) *is integrable and each of its leaves is a totally geodesic submanifold of* M*, namely* $\nabla_X Y \in V^0_{\lambda_i}$ *for all vectors* X *and* Y *of any* $V^0_{\lambda_i}$*.*

We here call $V^0_{\lambda_i} = \{X \in T^0M | AX = \lambda_i X\}$ the *restricted principal distribution* associated to the principal curvature λ_i.

We here again explain the property (iii) in the Introduction with proof.

Proposition 1. *In a nonflat complex space form* $\widetilde{M}_n(c), n \geqq 2$*, there exist no real hypersurfaces* M *with parallel structure tensor* ϕ*.*

Proof. Suppose there exists a real hypersurface with parallel structure tensor ϕ in a nonflat complex space form $\widetilde{M}_n(c), n \geqq 2$. Then it follows from (2.4) that

$$\eta(Y)AX - g(AX, Y)\xi = 0 \quad \text{for } \forall X, Y \in TM. \tag{2.35}$$

Putting $X = Y = \xi$ in (2.35), we get $A\xi = g(A\xi, \xi)\xi$, so that our real hypersurface M is a Hopf hypersurface. Next, we take a principal curvature vector X orthogonal to ξ with $AX = \lambda X$. Then, setting $Y = \xi$ in (2.35), we can see that the principal curvature vector X satisfies $AX = 0$, i.e., $\lambda = 0$. Hence our real hypersurface M is a Hopf hypersurface having two distinct constant principal curvature δ and $\lambda = 0$. However there does not exist such a Hopf hypersurface (see the above tables of principal curvatures), which is a contradiction. Therefore we obtain the desired statement. □

3. Classification of homogenenous Hopf hypersurfaces in $\mathbb{C}P^n(c)$ and $\mathbb{C}H^n(c)$

Theorem 1 ([11, 18])**.** *In* $\mathbb{C}P^n(c)$ $(n \geqq 2)$*, every Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following*:

- (A_1) *A geodesic sphere of radius* r*, where* $0 < r < \pi/\sqrt{c}$;
- (A_2) *A tube of radius* r *around a totally geodesic* $\mathbb{C}P^\ell(c)$ $(1 \leqq \ell \leqq n-2)$*, where* $0 < r < \pi/\sqrt{c}$;
- (B) *A tube of radius* r *around a complex hyperquadric* $\mathbb{C}Q^{n-1}$*, where* $0 < r < \pi/(2\sqrt{c})$;
- (C) *A tube of radius* r *around the Segre embbeding* $\mathbb{C}P^1(c) \times \mathbb{C}P^{(n-1)/2}(c)$*, where* $0 < r < \pi/(2\sqrt{c}\,)$ *and* n $(\geqq 5)$ *is odd*;
- (D) *A tube of radius* r *around a complex Grassmann* $\mathbb{C}G_{2,5}$*, where* $0 < r < \pi/(2\sqrt{c}\,)$ *and* $n = 9$;
- (E) *A tube of radius* r *around a Hermitian symmetric space* $SO(10)/U(5)$*, where* $0 < r < \pi/(2\sqrt{c}\,)$ *and* $n = 15$.

These real hypersurfaces are said to be of types (A_1), (A_2), (B), (C), (D) and (E). Summing up real hypersurfaces of types (A_1) and (A_2), we call them hypersurfaces of type (A). The numbers of distinct principal curvatures of these real hypersurfaces are $2, 3, 3, 5, 5, 5$, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}P^n(c)$ are given as follows (cf. [17]):

	(A_1)	(A_2)	(B)	(C, D, E)
λ_1	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r)$	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r)$	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r-\frac{\pi}{4})$	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r-\frac{\pi}{4})$
λ_2	—	$-\frac{\sqrt{c}}{2}\tan(\frac{\sqrt{c}}{2}r)$	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r+\frac{\pi}{4})$	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r+\frac{\pi}{4})$
λ_3	—	—	—	$\frac{\sqrt{c}}{2}\cot(\frac{\sqrt{c}}{2}r)$
λ_4	—	—	—	$-\frac{\sqrt{c}}{2}\tan(\frac{\sqrt{c}}{2}r)$
α	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$	$\sqrt{c}\cot(\sqrt{c}\,r)$

The multiplicities of these principal curvatures are given as follows (cf. [17]):

	(A_1)	(A_2)	(B)	(C)	(D)	(E)
$m(\lambda_1)$	$2n-2$	$2n-2\ell-2$	$n-1$	2	4	6
$m(\lambda_2)$	—	2ℓ	$n-1$	2	4	6
$m(\lambda_3)$	—	—	—	$n-3$	4	8
$m(\lambda_4)$	—	—	—	$n-3$	4	8
$m(\alpha)$	1	1	1	1	1	1

Theorem 2 ([5]). *In $\mathbb{C}H^n(c)$ $(n \geqq 2)$, every Hopf hypersurface all of whose principal curvatures are constant is locally congruent to one of the following homogeneous real hypersurfaces (see* [5, 17]*):*

- (A_0) *A horosphere in $\mathbb{C}H^n(c)$;*
- $(A_{1,0})$ *A geodesic sphere of radius r $(0 < r < \infty)$;*
- $(A_{1,1})$ *A tube of radius r around a totally geodesic $\mathbb{C}H^{n-1}(c)$, where $0 < r < \infty$;*
- (A_2) *A tube of radius r around a totally geodesic $\mathbb{C}H^{\ell}(c)$ $(1 \leqq \ell \leqq n-2)$, where $0 < r < \infty$;*
- (B) *A tube of radius r around a totally real totally geodesic $\mathbb{R}H^n(c/4)$, where $0 < r < \infty$.*

These real hypersurfaces are said to be of types (A_0), (A_1), (A_1), (A_2) and (B). Here, type (A_1) means either type $(A_{1,0})$ or type $(A_{1,1})$. Summing up real hypersurfaces of types (A_0), (A_1) and (A_2), we call them hypersurfaces of type (A). The homogeneous real hypersurface of type (B) with radius $r = (1/\sqrt{|c|}\,)\log_e(2+\sqrt{3}\,)$ has two distinct constant principal curvatures $\lambda_1 = \delta = \sqrt{3|c|}\,/2$ and $\lambda_2 = \sqrt{|c|}\,/(2\sqrt{3}\,)$. Except for this real hypersurface, the numbers of distinct principal curvatures of Hopf hypersurfaces with constant principal curvatures are $2, 2, 2, 3, 3$, respectively. The principal curvatures of these real hypersurfaces in $\mathbb{C}H^n(c)$ are given as follows :

	(A_0)	$(\mathrm{A}_{1,0})$	$(\mathrm{A}_{1,1})$	(A_2)	(B)
λ_1	$\frac{\sqrt{\lvert c\rvert}}{2}$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\coth(\frac{\sqrt{\lvert c\rvert}}{2}r)$
λ_2	—	—	—	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$	$\frac{\sqrt{\lvert c\rvert}}{2}\tanh(\frac{\sqrt{\lvert c\rvert}}{2}r)$
α	$\sqrt{\lvert c\rvert}$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\coth(\sqrt{\lvert c\rvert}\,r)$	$\sqrt{\lvert c\rvert}\tanh(\sqrt{\lvert c\rvert}\,r)$

The multiplicities of these principal curvatures are given as follows (cf. [5]):

	(A_0)	$(\mathrm{A}_{1,0})$	$(\mathrm{A}_{1,1})$	(A_2)	(B)
$m(\lambda_1)$	$2n-2$	$2n-2$	$2n-2$	$2n-2\ell-2$	$n-1$
$m(\lambda_2)$	—	—	—	2ℓ	$n-1$
$m(\alpha)$	1	1	1	1	1

We here explain briefly the discussion in the proof of Theorem 2. The discussion is divided into two parts. The first part is to show that a Hopf hypersurface M with constant principal curvatures has at most three distinct principal curvatures. The second part is to show the classification of Hopf hypersurface M with constant principal curvatures.

In both parts, the following Lemma 6 is very crucial. To get Lemma 6, we prepare Lemma 5.

Lemma 5 ([5]). *Let $E_1,\dots,E_{2n-2}$ a local field of orthonormal frames of T^0M with $AE_i=\lambda_iE_i$ $(1\leqq i\leqq 2n-2)$. Then for every $i\in\{1,\dots,2n-2\}$ we have*

$$\sum_{\substack{j=1\\ \lambda_j\neq\lambda_i}}^{2n-2}\frac{(c/4)+\lambda_i\lambda_j}{\lambda_i-\lambda_j}\big(1+2g(\phi E_i,E_j)^2\big)=0.$$

By virtue of Lemma 5, we have the following Lemma 6 which is a special case of Lemma 1.

Lemma 6 ([5]). (1) *If $\delta^2+c=0$, then $Spec(T^0M)=\{\delta/2\}$; in particular we have two distinct constant principal curvatures in this case.*
(2) *If $\delta^2+c\neq 0$, then for all $\lambda\in Spec(T^0M)$ we have $2\lambda-\delta\neq 0$ and $JV^0_\lambda=V^0_{\lambda^*}$, where $\lambda^*\in Spec(T^0M)$ is uniquely characterized by the equation $(2\lambda-\delta)(2\lambda^*-\delta)=\delta^2+c$.*

In the first part of the proof, by Lemma 5 and Lemma 6, we obtain the following theorem:

Theorem 3 ([5]). *For every $\lambda\in Spec(T^0M)$ there exists exactly one $\lambda^*\in Spec(T^0M)$ such that $JV^0_\lambda=V^0_{\lambda^*}$ and*

$$\sum_{\substack{\mu\in Spec(T^0M)\\ \mu\neq\lambda}} m_\mu\frac{(c/4)+\lambda\mu}{\lambda-\mu}+Q(\lambda^*)=0,$$

where m_μ is the multiplicity of the eigenvalue μ with respect to $A|_{T^0M}$ and

$$Q(\lambda^*) = \begin{cases} 0 & \text{if } \lambda = \lambda^*, \\ 2\,\dfrac{(c/4) + \lambda\lambda^*}{\lambda - \lambda^*} & \text{if } \lambda \neq \lambda^*. \end{cases}$$

Due to Theorem 3, in the case $\lambda, \mu \in \mathrm{Spec}(T^0M)$, $\lambda \neq \mu$, it is shown that $(c/4) + \lambda\mu = 0$. Therefore, M has at most three distinct principal curvatures.

In the second part, using a geodesic variation, we can obtain the classification of Hopf hypersurfaces M all of whose principal curvatures are constant.

The rough sketch of the second part is as follows. Here, we set $c = -4$. For $p \in M$, let $\gamma_p : [0, \infty) \to \mathbb{C}H^n$ be the unique geodesic of $\mathbb{C}H^n$ with $\dot{\gamma}_p(0) = \mathcal{N}_p$, where $\mathcal{N}_p$ is the unit normal vector of M at the point p. For a fixed $r \in (0, \infty)$, a map $\Phi : M \to \mathbb{C}H^n$ is defined by $\Phi(p) := \gamma_p(r)$. Moreover, for $v \in T_pM$, let B_v be the unique parallel field along γ_p with $B_v(0) = v$ and X_v the Jacobi field along γ_p with initial condition that $X_v(0) = v$ and $X'_v(0)(= \widetilde{\nabla}_{\dot{\gamma}_p(0)} X_v) = -Av$, where $\widetilde{\nabla}$ is the covariant differentiation on the ambient space $\mathbb{C}H^n$ and A is the shape operator of M in $\mathbb{C}H^n$. We take a curve c in M satisfying that $\dot{c}(0) = v$ and the variation field of the variation $V(t,s) := \exp^{\mathbb{C}H^n}(t\mathcal{N} \circ c(s))$ is X_v, namely $X_v = \dfrac{\partial}{\partial s} V(t,s)\Big|_{s=0}$. By the Jacobi equation and (2.7), X_v is expressed in the following form.

$$X_v(t) = (\cosh(t) - \lambda \sinh(t)) B_v(t) \quad \text{for } v \in V^0_\lambda(p),\ \lambda \in \mathrm{Spec}(T^0M), \tag{3.1}$$

$$X_v(t) = \Big(\cosh(2t) - \frac{\delta}{2}\sinh(2t)\Big) B_v(t) \quad \text{for } v \in \mathbb{R}\xi_p. \tag{3.2}$$

We can get the following well-known formulas.

$$X_v(r) = (\Phi_*)_p v, \tag{3.3}$$

$$X'_v(r) = \widetilde{\nabla}_{X_v(r)} \dot{\gamma}_p = \widetilde{\nabla}^\Phi_v \mathcal{E}, \tag{3.4}$$

where $\widetilde{\nabla}^\Phi$ is the pull-back connection on $\Phi^{-1}T\mathbb{C}H^n$ by Φ.

Thinking about the case when ϕ is not immersion, i.e., ϕ_* is not injective, we know the following lemma.

Lemma 7 ([5])**.** *If Φ is not an immersion, i.e. $\coth(r) \in \mathrm{Spec}(T^0M)$ or $\delta = 2\coth(2r)$, then $\Phi(M)$ consists only of focal points, and for every $p_0 \in M$ there exists a neighborhood V in M such that $\Phi|_V$ is a submersion onto a regular submanifold W of $\mathbb{C}H^n$. Furthermore, the spectrum of the shape operator $\tilde{A}_z$ of W does not depend on $z \in N_1W$, which is the unit normal bundle along W, and it is given by*

$$\Sigma := \begin{cases} \{\tilde{\lambda} | \lambda \in \mathrm{Spec}(T^0M), \lambda \neq \coth(r)\}, & \text{if } \delta = 2\coth(2r), \\ \{\tilde{\delta}\} \cup \{\tilde{\lambda} | \lambda \in \mathrm{Spec}(T^0M), \lambda \neq \coth(r)\}, & \text{if } \delta \neq 2\coth(2r), \end{cases}$$

where

$$\tilde{\lambda} := \frac{-\sinh(r) + \lambda\cosh(r)}{\cosh(r) - \lambda\sinh(r)} \quad \text{if } \lambda \neq \coth(r),$$

and

$$\tilde{\delta} := 2\,\frac{-2\sinh(2r) + \delta\cosh(2r)}{2\cosh(2r) - \delta\sinh(2r)} \quad \text{if } \delta \neq 2\coth(2r).$$

In the proof of Lemma 7, because Φ has constant rank for each r, to show the existence of the neighborhood V, implicit function theorem is used. From Lemma 7, the classification of Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n(c)$ $(n \geqq 2)$ follows.

For example, when the real hypersurface M has three distinct principal curvatures $\lambda = \tanh(r)$, $\mu = \coth(r)$, $\delta = 2\tanh(2r)$ and $JV_\lambda^0 = V_\mu^0$, we can say that M is locally congruent to a tube of radius r around a totally real totally geodesic submanifold $\mathbb{R}H^n$ of $\mathbb{C}H^n$ of real dimension n.

4. EXTRINSIC SHAPES OF GEODESICS

From the viewpoint of extrinsic shapes of geodesics on a connected hypersurface M^n in the ambient space $\widetilde{M}^{n+1}$, the following proposition is fundamental.

Proposition 2. *For a connected hypersurface M^n isometrically immersed into a Riemannian manifold $\widetilde{M}^{n+1}$, the following three conditions are mutually equivalent*:

1. *Every geodesic on M^n is mapped to a circle in $\widetilde{M}^{n+1}$*;
2. *Every geodesic on M^n is mapped to a circle of the same curvature in $\widetilde{M}^{n+1}$*;
3. *M^n is totally umbilic in $\widetilde{M}^{n+1}$ and M^n has constant mean curvature, namely* Trace A *is constant on M^n, where A is the shape operator of M^n in $\widetilde{M}^{n+1}$.*

It is known that in $\widetilde{M}_n(c)(c \neq 0, n \geqq 2)$ there does not exist a totally umbilic real hypersurface M^{2n-1}. Therefore, by Propsition 2 there does not exist a real hypersurface M^{2n-1} all of whose geodesics are mapped to circles in the ambient space $\widetilde{M}_n(c)$.

Proof of Proposition 1. We suppose Condition (1). By the definition of a circle in a Riemannian manifold, every geodesic γ of M^n, considered as a curve in the ambient space $\widetilde{M}^{n+1}$, satisfies the equation $\widetilde{\nabla}_{\dot\gamma}\dot\gamma = kY_s$ and $\widetilde{\nabla}_{\dot\gamma}Y_s = -k\dot\gamma$ with a non-negative constant k and a unit vector Y_s along γ. This equation can be transformed to the following ordinary differential equation:

$$\widetilde{\nabla}_{\dot\gamma}\widetilde{\nabla}_{\dot\gamma}\dot\gamma + g(\widetilde{\nabla}_{\dot\gamma}\dot\gamma, \widetilde{\nabla}_{\dot\gamma}\dot\gamma)\dot\gamma = 0. \tag{4.1}$$

Making use of Gauss formula: $\widetilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)\mathcal{N}$ and Weingarten formula: $\widetilde{\nabla}_X \mathcal{N} = -AX$ for the hypersurface M^n in $\widetilde{M}^{n+1}$, we can rewrite (4.1) as follows:

$$-g(A\dot\gamma, \dot\gamma)A\dot\gamma + g(A\dot\gamma, \dot\gamma)^2\dot\gamma + g((\nabla_{\dot\gamma}A)\dot\gamma, \dot\gamma)\mathcal{N} = 0. \tag{4.2}$$

Therefore, taking the tangential component and the normal component of (4.2) for the hypersurace M^n in $\widetilde{M}^{n+1}$, we obtain

$$g(A\dot\gamma, \dot\gamma)A\dot\gamma = g(A\dot\gamma, \dot\gamma)^2\dot\gamma \quad \text{and} \quad g((\nabla_{\dot\gamma}A)\dot\gamma, \dot\gamma) = 0 \tag{4.3}$$

for each geodesic γ on M^n. We can say Equation (4.3) means that

$$g(AX, X)AX = g(AX, X)^2X \quad \text{and} \quad g((\nabla_X A)X, X) = 0 \tag{4.4}$$

for all $X \in TM$ with $\|X\| = 1$. Note that the former equation in (4.4) means

$$g(AX, X)g(AX, Y) = 0 \tag{4.5}$$

for each pair of orthonormal vectors X and Y on M, which is equivalent to saying that

$$g(A_pX,X)^2 \text{ is constant at each point } p \in M \tag{4.6}$$

for every unit vector $X \in T_pM$ as shown below .

Indeed, let $f : S^{n-1}(1)(\subset \mathbb{R}^n) \to \mathbb{R}$ be the differentiable function on a subset $S^{n-1}(1) \cong \{u \in T_pM | \|u\| = 1\}$ defined by $f(u) = g(A_pu,u)^2$, where A_p is the shape operator of M in $\widetilde{M}^{n+1}$ at the point $p \in M$. If v is a vector tangent to $S^{n-1}(1)$ at u (hence $u \perp v$), we find $v(f) = 4g(A_pu,u)g(A_pu,v) = 0$ by (4.5). Here we use the fact $\widetilde{\nabla}_v u = v$, where $\widetilde{\nabla}$ is a Riemannian connection of $\mathbb{R}^n$ and u is the position vector at the point $u \in S^{n-1}(1)$. Thus f is a constant function on $S^{n-1}(1)$. Then we can set $\lambda^2(p) = g(AX,X)^2$ for each unit vector $X \in T_pM$ with $\lambda(p) \geqq 0$ at every point $p \in M$. When M^n is not totally geodesic in $\widetilde{M}^{n+1}$, there exists a point $x \in M$ with $\lambda(x) > 0$. Then the continuity of the function λ shows that there exists some open neighborhood U_x of the point x such that $\lambda > 0$ on U_x. We here choose a local field of orthonormal frames $e_1, \ldots, e_n$ on U_x in such a way that $Ae_i = \lambda_i e_i$ $(1 \leqq i \leqq n)$. Hence, from (4.6) we see that $\lambda_1^2 = \ldots = \lambda_n^2 = \lambda^2$. We here suppose that there exists an orthonormal pair of vectors e_i and e_j such that $Ae_i = \lambda e_i$ and $Ae_j = -\lambda e_j$. Then we find that

$$g(A(e_i+e_j)/\sqrt{2}, (e_i+e_j)/\sqrt{2}\,) = 0,$$

which is a contradiction. So, we know that either $Ae_i = \lambda e_i$ $(1 \leqq i \leqq n)$ or $Ae_i = -\lambda e_i$ $(1 \leqq i \leqq n)$, which shows that every point $y \in U_x$ is an umbilic point. Thus we can see that M^n is totally umbilic in $\widetilde{M}^{n+1}$. Furthermore, the latter equation in (4.4) yields that the function λ is constant on M. Therefore we get Conditions (2) and (3) in our Proposition.

By virtue of the above argument we can see that each of Conditions (2) and (3) implies Condition (1). $\square$

Weakening the definition of a totally umbilic real hypersurfaces, that of a totally η-umbilic real hypersurface is given as follows.

Definition. *A real hypersurface M^{2n-1} is totally η-umbilic in $\widetilde{M}_n(c)$ if the shape operator A is written as $AX = \alpha X + \beta\eta(X)\xi$ for all $X \in TM$, where α, β are functions on M.*

It is a well known fact that, every totally η-umbilic hypersurface is locally congruent to one of types $(\mathrm{A}_1)(c>0)$ and types $(\mathrm{A}_0), (\mathrm{A}_{1,0}), (\mathrm{A}_{1,1})(c<0)$ and α, β are automatically constant on M.

5. Homogeneous real hypersurfaces of type (B) in $\mathbb{C}H^n(c)$

Our aim here is to prove the following:

Theorem ([21]). *Let M be a connected real hypersurface of $\mathbb{C}H^n(c), n \geqq 2$. Then the following conditions* (1), (2), (3) *are mutually equivalent.*

(1) *M is locally congruent to the homogeneous real hypersurface of type (B) with two distinct principal curvatures in $\mathbb{C}H^n(c)$.*
(2) *M satisfies the following two conditions $2_a)$ and $2_b)$.*

2_a) *The exterior differentiation $d\eta$ of the contact form η on M which is given by $d\eta(X,Y) := (1/2)\{X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y])\}$ holds either $d\eta(X,Y) = (\sqrt{|c|/3}\,)g(X,\phi Y)$ for all $X,Y \in TM$ or $d\eta(X,Y) = (-\sqrt{|c|/3}\,)g(X,\phi Y)$ for all $X,Y \in TM$.*

2_b) *There exist two geodesics $\gamma_i = \gamma_i(s)$ $(i = 1,2)$ on M through a point $x = \gamma_1(0) = \gamma_2(0)$ with initial vectors $\dot\gamma_i(0)$ orthogonal to $\xi_{\gamma_i(0)}$ which are mapped to circles of different positive curvatures.*

(3) *M satisfies the following three conditions $3_a), 3_b)$ and $3_c)$.*

3_a) *The holomorphic distribution $T^0M := \{X \in TM | X \perp \xi\}$ of M is decomposed as the direct sum of restricted principal distributions $V^0_{\lambda_i} = \{X \in T^0M | AX = \lambda_i X\}$.*

3_b) *The derivative of the shape operator A of M satisfies $(\nabla_X A)Y = 0$ for all vectors X,Y of each $V^0_{\lambda_i}$ in 3_a).*

3_c) *There exist two geodesics $\gamma_i = \gamma_i(s)$ $(i = 1,2)$ on M through a point $x = \gamma_1(0) = \gamma_2(0)$ with initial vectors $\dot\gamma_i(0)$ orthogonal to $\xi_{\gamma_i(0)}$ which are mapped to circles of positive curvatures $3k$ and k, respectively.*

Proof. We first prove that Condition (1) implies both Conditions (2) and (3). Let M be the homogeneous real hypersurface of type (B) with two distinct principal curvatures. Then our real hypersurface M satisfies (see Lemma 1 (1))

$$A\xi = \lambda_1\xi,\ T^0M = V^0_{\lambda_1} \oplus V^0_{\lambda_2} \quad \text{and} \quad \phi V^0_{\lambda_1} = V^0_{\lambda_2},$$

where $\lambda_1 = \sqrt{3|c|}/2$ and $\lambda_2 = \sqrt{|c|}/(2\sqrt{3}\,)$. Hence we easily see that

$$(\phi A + A\phi)X = \frac{2\sqrt{|c|}}{\sqrt{3}}\phi X \quad \text{for all } X \in TM. \tag{5.1}$$

It follows from (2.5) and (5.1) that

$$\begin{aligned} d\eta(X,Y) &= \frac{1}{2}(g(Y,\nabla_X\xi) - g(X,\nabla_Y\xi)) \\ &= \frac{1}{2}g((\phi A + A\phi)X,Y) = \frac{\sqrt{|c|}}{\sqrt{3}}g(\phi X,Y) \\ &= -\frac{\sqrt{|c|}}{\sqrt{3}}g(X,\phi Y) \end{aligned}$$

Needless to say, when we take a unit normal vector $\mathcal{N}$ with $A\xi = -\lambda_1\xi$, we get $d\eta(X,Y) = (\sqrt{|c|}\sqrt{3})g(X,\phi Y)$ for all $X,Y \in TM$

We next take two geodesics $\gamma_1 = \gamma_1(s)$ and $\gamma_2 = \gamma_2(s)$ on M through an arbitrary fixed point $x = \gamma_1(0) = \gamma_2(0)$ with initial vectors $\dot\gamma_1(0) \in V^0_{\lambda_1}$ and $\dot\gamma_2(0) \in V^0_{\lambda_2}$, respectively. Then by virtue of Lemma 4 we find that these curves γ_1 and γ_2 can be considered as geodesics on some leaves L_{λ_1} and L_{λ_2} of the restricted principal distributions $V^0_{\lambda_1}$ and $V^0_{\lambda_2}$, respectively. We here explain these leaves L_{λ_1} and L_{λ_2} in detail. Due to (2.1) we see that the L_{λ_1} and L_{λ_2} are totally umbilic hypersurfaces of an n-dimensional totally real totally geodesic real hyperbolic space $\mathbb{R}H^n(c/4)$ of constant sectional curvature $c/4$ in $\mathbb{C}H^n(c)$. Hence these leaves are locally congruent to real space forms $M^{n-1}(d_i)$ of constant sectional curvatures d_i $(i = 1,2)$, respectively. So we have the equations $d_i - (c/4) = \lambda_i^2$. Thus we know that L_{λ_1} and

L_{λ_2} are locally congruent to real space forms $M^{n-1}(|c|/2)$ of constant sectional curvature $|c|/2$ and $M^{n-1}(c/6)$ of constant sectional curvature $c/6$, respectively. This, together with Equation (2.1), shows that these geodesics γ_1 and γ_2 are mapped to circles of different positive curvatures λ_1 and λ_2 in $\mathbb{C}H^n(c)$, respectively. Thus we obtain Condition (2).

The above discussion yields that Conditions 3_a) and 3_c) are immediate consequences of Condition (1). So we here verify Condition 3_b). Thanks to Lemma 4 our real hypersurface M satisfies $\nabla_X Y \in V^0_{\lambda_i}$ for all X, Y of the restricted principal distributions $V^0_{\lambda_i}$ $(i = 1, 2)$. For all $X, Y \in V^0_{\lambda_i}$ $(i = 1, 2)$ with $\lambda_1 = \sqrt{3|c|}/2$ and $\lambda_2 = \sqrt{|c|}/(2\sqrt{3}\,)$, we have

$$\begin{aligned}(\nabla_X A)Y &= \nabla_X(AY) - A\nabla_X Y\\ &= \lambda_i\nabla_X Y - \lambda_i\nabla_X Y = 0,\end{aligned}$$

so that we get Condition 3_b). Therefore we can see that Condition (1) implies both Conditions (2) and (3).

Conversely, we suppose Condition (2). Then, from Condition 2_a) and the above discussion we can take a unit normal vector $\mathcal{N}$ on M satisfying (5.1). Setting $X = \xi$ in (5.1), we get $\phi A\xi = 0$, so that our real hypersurface M is a Hopf hypersurface. Next we take a principal curvature vector field $X(\perp \xi)$ with a principal curvature λ in (5.1). Suppose that $2\lambda - \delta \neq 0$ at some point $x \in M$. Hence, from the continuity of the function $2\lambda - \delta$ there exists a sufficiently small neighborhood $\mathcal{U}_x$ of the point x satisfying that $(2\lambda - \delta)(y) \neq 0$ for each $y \in \mathcal{U}_x$ This, combined with Lemma 1(1) and (5.1), yields the following equation on the neighborhood $\mathcal{U}_x$:

$$\lambda + \frac{\delta\lambda + \frac{c}{2}}{2\lambda - \delta} = \frac{2\sqrt{|c|}}{\sqrt{3}}.$$

Thus, from the constancy of δ and this equation we find that λ is constant on $\mathcal{U}_x$.

We finally consider the case of $2\lambda - \delta = 0$ at some point of M. We shall verify that $2\lambda - \delta$ vanishes identically on M. Assume that $2\lambda - \delta \neq 0$ at some point $x_0 \in M$, and set $y_0 = (2\lambda - \delta)(x_0)$. Let N be the subset of those points $x \in M$ such that $(2\lambda - \delta)(x) = y_0$. Clearly N is a non-empty closed subset of M. It is also open, since the discussion in the case of $2\lambda - \delta \neq 0$ means that the function $2\lambda - \delta$ is constantly equal to $y_0 \neq 0$ on some neighborhood of each point $x \in N$. Since M is connected, we find that $N = M$, which is a contradiction. So we find that $\lambda = \delta/2$ on M.

Thus we can see that our real hypersurface is a Hopf hypersurface with constant principal curvatures. In consideration of Theorem 2 we see that M is of type either (A_0), (A_1), (A_2) or (B). We shall check 2_a) for these real hypersurfaces one by one. For the properties of the principal curvatures of real hypersurfaces of type (A), see [17].

Let M be of type (A_0). Then $AX = (\sqrt{|c|}/2)X$ for all $X \in T^0M$ and $A\xi = \sqrt{|c|}\,\xi$, which implies $(\phi A + A\phi)X = \sqrt{|c|}\,\phi X$ for all $X \in TM$. So we have $d\eta(X,Y) = (-\sqrt{|c|}/2)g(X, \phi Y)$ for all $X, Y \in TM$, which is a contradiction. Thus we can see that this real hypersurface does not satisfy Condition 2_a).

Let M be of type ($A_{1,0}$). Then $AX = (\sqrt{|c|}/2)\coth(\sqrt{|c|}\,r/2)X$ for all $X \in T^0M$ and $A\xi = \sqrt{|c|}\,\coth(\sqrt{|c|}\,r)\xi$. Hence, by the same computation as above

$d\eta(X,Y) = (-\sqrt{|c|}/2)\coth(\sqrt{|c|}\,r/2)g(X,\phi Y)$. Thus we obtain the equation $\sqrt{|c|}\,\coth(\sqrt{|c|}\,r/2) = 2\sqrt{|c|}/\sqrt{3}$. Solving this equation, we know that the radius r of real hypersurfaces of type ($\mathrm{A}_{1,0}$) is expressed as $r = (2/\sqrt{|c|}\,)\log_e(2+\sqrt{3}\,)$. Then this real hypersurface satisfies Condition 2_a).

Let M be of type ($\mathrm{A}_{1,1}$). Then $AX = (\sqrt{|c|}/2)\tanh(\sqrt{|c|}\,r/2)X$ for all $X \in T^0M$ and $A\xi = \sqrt{|c|}\,\coth(\sqrt{|c|}\,r)\xi$. Hence we get $d\eta(X,Y) = (-\sqrt{|c|}/2)\cdot$ $\tanh(\sqrt{|c|}\,r/2)g(X,\phi Y)$, so that $\sqrt{|c|}\,\tanh(\sqrt{|c|}\,r/2) = 2\sqrt{|c|}/\sqrt{3}$, which is a contradiction. Thus this real hypersurface does not satisfy Condition 2_a).

Let M be of type (A_2). Then the holomorphic distribution T^0M of M is decomposed as $T^0M = V^0_{\lambda_1} \oplus V^0_{\lambda_2} = V_{\lambda_1} \oplus V_{\lambda_2}$ with $\lambda_1 = (\sqrt{|c|}\,)\coth(\sqrt{|c|}\,r/2)$, $\lambda_2 = (\sqrt{|c|}\,)\tanh(\sqrt{|c|}\,r/2)$ and $A\xi = \sqrt{|c|}\,\coth(\sqrt{|c|}\,r)\xi$. Note that $\phi V^0_{\lambda_1} = V^0_{\lambda_1}$ and $\phi V^0_{\lambda_2} = V^0_{\lambda_2}$. These equalities imply that Equation (5.1) does *not* hold. Hence this real hypersurface does not hold Condition 2_a).

Let M be of type (B). Then it follows from (5.1) that

$$\frac{\sqrt{|c|}}{2}\coth\Big(\frac{\sqrt{|c|}\,r}{2}\Big) + \frac{\sqrt{|c|}}{2}\tanh\Big(\frac{\sqrt{|c|}\,r}{2}\Big) = \frac{2\sqrt{|c|}}{\sqrt{3}}.$$

Solving this equation, we get $r = (1/\sqrt{|c|}\,)\log_e(2+\sqrt{3}\,)$, so that M has two distinct principal curvatures $\lambda_1 = \delta = \sqrt{3|c|}/2$ and $\lambda_2 = \sqrt{|c|}/(2\sqrt{3}\,)$, which yields that this real hypersurface satisfies Condition 2_a). Hence our discussion asserts that a real hypersurface of $\mathbb{C}H^n(c)$ satisfies Condition 2_a) if and only if M is locally congruent to either the geodesic sphere $G(r)$ of radius $r = (2/\sqrt{|c|}\,)\log_e(2+\sqrt{3}\,)$ (i.e., $(\sqrt{|c|}/2)\coth(\sqrt{|c|}\,r/2) = \sqrt{|c|}/\sqrt{3}\,$) or the real hypersurface of type (B) with two distinct principal curvatures. However the former case does not satisfy Condition 2_b). Indeed, every geodesic $\gamma = \gamma(s)$ on $G(r)$ of radius $r = (2/\sqrt{|c|}\,)\log_e(2+\sqrt{3}\,)$ with initial vector $\dot\gamma(0)$ orthogonal to $\xi_{\gamma(0)}$ is mapped to a circle of the same curvature $\sqrt{|c|}/\sqrt{3}$ (cf. [7]). Therefore we can see that Condition (2) implies Condition (1).

We finally suppose Condition (3). We shall study on the open dense subset $\mathcal{U}$ of M, which is given by Remark 2. Condition 3_a) implies that our real hypersurface M is a Hopf hypersurface. Next, for each vector X, Y of any restricted principal distribution $V^0_{\lambda_i}$, from Condition 3_b) we get

$$\begin{aligned} 0 &= (\nabla_X A)Y = \nabla_X(AY) - A\nabla_X Y \\ &= \nabla_X(\lambda_i Y) - A\nabla_X Y \\ &= (X\lambda_i)Y + (\lambda_i I - A)\nabla_X Y, \end{aligned}$$

so that

$$(X\lambda_i)Y + (\lambda_i I - A)\nabla_X Y = 0. \tag{5.2}$$

We here take an arbitrary vector $Z \in V^0_{\lambda_j}$ with $\lambda_i \neq \lambda_j$.

Note that there exists such a vector Z. To show that, we assume that there does not exist such Z. Then there exists an open set $\mathcal{V}$ of M such that $T^0\mathcal{V} = V^0_{\lambda_i}$, which means that $Av = \lambda_i v$ for any $v \in T^0\mathcal{V}$, i.e., our real hypersurface $\mathcal{V}$ is totally η-umbilic in the ambient space $\mathbb{C}H^n(c)$. Hence this real hypersurface $\mathcal{V}$ is of type

(A). So the shape operator A of $\mathcal{V}$ satisfies the following differential equation

$$(\nabla_X A)Y = (-c/4)(g(\phi X, Y)\xi + \eta(Y)\phi X)$$

for $X, Y \in T\mathcal{V}$ (see [17]). Then for any unit vector X orthogonal to ξ, from the above equation we have $(\nabla_X A)\phi X = (-c/4)\xi \neq 0$, which contradicts to Condition 3_b).

Then taking the inner product of the left hand side of (5.2) and the vector $Z \in V^0_{\lambda_j}$ with $\lambda_i \neq \lambda_j$, we obtain

$$0 = g((\lambda_i I - A)\nabla_X Y, Z) = (\lambda_i - \lambda_j)g(\nabla_X Y, Z),$$

so that $g(\nabla_X Y, Z) = 0$, which shows that

$$\nabla_X Y \in V^0_{\lambda_i} \oplus \{\xi\}_{\mathbb{R}} \quad \text{for each } X, Y \in V^0_{\lambda_i}. \tag{5.3}$$

We shall verify that $g(\nabla_X Y, \xi) = 0$. It follows from (2.5) that

$$g(\nabla_X Y, \xi) = -g(Y, \nabla_X \xi) = -g(Y, \phi AX) = -\lambda_i g(Y, \phi X). \tag{5.4}$$

On the other hand, from (2.9) and Condition 3_b) we find that $g(\phi X, Y) = 0$ for each $X, Y \in V^0_{\lambda_i}$. This, together with (5.3) and (5.4), yields that $\nabla_X Y \in V^0_{\lambda_i}$ for every vector X, Y of any restricted principal distribution $V^0_{\lambda_i}$. Hence our real hypersurface M is of type (B) on the open dense subset $\mathcal{U}$ (see Lemma 4), which implies that M is globally of type (B).

The rest of the proof is to determine real hypersurfaces of type (B) satisfying Condition 3_c).

To do this, we review the following fact. We take a geodesic $\gamma = \gamma(s)$ on a hypersuface M^n isometrically immersed into an $(n+1)$-dimensional Riemannian manifold $\widetilde{M}^{n+1}$ (with Riemannian metric g). Suppose that the geodesic γ is mapped to a circle of positive curvature k in the ambient space $\widetilde{M}^{n+1}$. Then the shape operator A of M^n in $\widetilde{M}^{n+1}$ satisfies

$$A\dot{\gamma}(s) = k\dot{\gamma}(s) \quad \text{for each } s \quad \text{or} \quad A\dot{\gamma}(s) = -k\dot{\gamma}(s) \quad \text{for each } s. \tag{5.5}$$

In fact, it follows from (2.1), (2.2) and the former equation in (4.3) that $g(A\dot{\gamma}, \dot{\gamma})A\dot{\gamma} = k^2\dot{\gamma}$ holds on the curve γ. This, combined with $k \neq 0$, yields (5.5).

In view of the above fact and Condition 3_c) we have only to consider the following equation

$$\frac{\sqrt{|c|}}{2}\coth\Big(\frac{\sqrt{|c|}\,r}{2}\Big) = 3\frac{\sqrt{|c|}}{2}\tanh\Big(\frac{\sqrt{|c|}\,r}{2}\Big).$$

Solving this equation, we see that $r = (1/\sqrt{|c|})\log_e(2+\sqrt{3}\,)$. Thus we obtain Condition (1). □

Remark 1. As an immediate consequence of the discussion in the proof of our Theorem we have the following:

Corollary. *A connected real hypersurface M of $\mathbb{C}H^n(c), n \geqq 2$ is of type (B) if and only if M satisfies the following two conditions*:

1. *The holomorphic distribution T^0M of M is decomposed as the direct sum of restricted principal distributions $V^0_{\lambda_i} = \{X \in T^0M | AX = \lambda_i X\}$;*
2. *The derivative of the shape operator A of M satisfies $(\nabla_X A)Y = 0$ for all vectors X, Y of each $V^0_{\lambda_i}$ in (1).*

Remark 2. As a matter of fact, if a real hypersurface M of $\mathbb{C}H^n(c)$ satisfies Condition (1) in the above Corollary, then M is a Hopf hypersurface. Note that the converse does not hold in general. However every Hopf hypersurface M^{2n-1} of $\mathbb{C}H^n(c)$ satisfies *locally* Condition (1) on an open dense subset

$$\mathcal{U} = \left\{ x \in M^{2n-1} \,\middle|\, \begin{array}{l} \text{the multiplicity of every principal curvature of } M^{2n-1} \text{ in} \\ \mathbb{C}H^n(c) \text{ is constant on some neighborhood } \mathcal{V}_x(\subset \mathcal{U}) \text{ of } x \end{array} \right\}$$

of M^{2n-1}.

Remark 3. If we remove Condition $2_b)$, our Theorem is not true. In fact, the geodesic sphere $G(r)$ with radius $r = (2/\sqrt{|c|}\,)\log_e(2+\sqrt{3}\,)$ in $\mathbb{C}H^n(c)$ satisfies Condition $2_a)$.

6. Characterization of real hypersurfaces of type (A)

The main result in this section is as follows:

Theorem 4. *Let $M^{2n-1}\,(n \geqq 2)$ be a connected real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$ through an isometric immersion. Then M is locally congruent to a hypersurface of type* (A) *if and only if the tensor $\psi = \phi A - A\phi$ is parallel, where ϕ and A are the structure tensor and the shape operator on M, respectively.*

Using the discussion in the proof of Theorem 4, we can prove Theorem 5.

Theorem 5. *Let M be a complex $n(\geqq 2)$-dimensional Kähler manifold. Then the following two conditions are mutually equivalent.*

1. *M is locally congruent to a complex space form.*
2. *At any point $m \in M$, the tensor $\psi_{m,r} = \phi_{m,r}A_{m,r} - A_{m,r}\phi_{m,r}$ on every sufficiently small geodesic sphere $G_m(r)$ of M is parallel in the direction of the characteristic vector $\xi_{m,r}$ of $G_m(r)$, i.e., $\nabla_{\xi_{m,r}}\psi_{m,r} = 0$, where $\phi_{m,r}$ and $A_{m,r}$ are the structure tensor and the shape operator on $G_m(r)$, respectively in the ambient space M.*

For the proof of Theorem 5, we prepare the following lemma due to Chen and Vanheche (see [8]).

Lemma 8. *Let M be a Riemannian manifold of dimension greater than two with Riemannian metric g. We denote by $G_m(r)$ a geodesic sphere with center m and radius r in M, and by $A_{m,r}$ the shape operator of $G_m(r)$ in M with respect to the outward unit normal vector field. For non-zero tangent vectors $v, w \in T_mM$ at a point $m \in M$, we choose a unit tangent vector $u \in T_mM$ orthogonal to both v and w. We denote by $v_r, w_r \in T_{\exp_m(ru)}M$ the parallel displacement of v, w along the geodesic segment $exp_m(su)$, $0 \leqq s \leqq r$. Then for sufficiently small r we have*

$$g(A_{m,r}v_r, w_r) = \frac{1}{r}\,g(v,w) + \frac{r}{3}\,g(R(u,v)w,u) + O(r^2), \tag{6.1}$$

where R is the Riemannian curvature tensor defined by $R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$.

7. Proof of Theorem 4

We first mention the following well-known proposition without proof (see [16, 17]).

Proposition 3. *Let M^{2n-1} $(n \geqq 2)$ be a connected real hypersurface in a nonflat complex space form $\widetilde{M}_n(c)$ through an isometric immersion. Then the following two conditions are mutually equivalent.*

(1) *M is locally congruent to a hypersurface of type* (A).
(2) *$\phi A = A\phi$, i.e., $\psi = \phi A - A\phi = 0$ on M.*

We first suppose that our real hypersurface M is locally congruent to a hypersurface of type (A). Then from Proposition 3, we know $\psi = 0$, so that ψ is parallel in a trivial sense.

Next, we suppose that ψ is parallel on M. The following discussion is essentially due to the work of [9]. By direct computation, from (2.4) we get the following:

$$\begin{aligned} 0 = g((\nabla_X\psi)Y, Z) &= \eta(AY)g(AX, Z) - \eta(Z)g(AX, AY) \\ &\quad + g((\phi(\nabla_X A) - (\nabla_X A)\phi)Y, Z) \\ &\quad + \eta(AZ)g(AX, Y) - \eta(Y)g(AX, AZ) \end{aligned} \tag{7.1}$$

for any tangent vector fields X, Y and Z on M. Putting $X = Y = Z = \xi$ in (7.1), because of (2.3) we obtain

$$g((\nabla_\xi\psi)\xi, \xi) = 2(\eta(A\xi)^2 - \eta(A^2\xi)).$$

So we have

$$\eta(A\xi)^2 - \eta(A^2\xi) = 0 \tag{7.2}$$

Setting $A\xi = a\xi + bU$ for two functions a and b on M, where U is the unit tangent vector field orthogonal to ξ on M and substituting this in (7.2), we get $a^2-(a^2+b^2) = 0$. So, we conclude that $b = 0$, i.e., ξ is a principal curvature vector. Namely, our real hypersurface is a Hopf hypersurface.

Next, we take a principal curvature vector X with principal curvature λ orthogonal to ξ. Setting $Y = X, Z = \xi$ in (7.1) and multiplying $2\lambda - \delta$ to (7.1), from Lemma 1, (2.3), (2.4), (2.5), we find

$$\begin{aligned} 0 &= (2\lambda - \delta)g((\nabla_X\psi)X, \xi) \\ &= (-\lambda^2 + \delta\lambda)(2\lambda - \delta)g(X, X) - (2\lambda - \delta)g(\phi X, (\nabla_X A)\xi) \\ &= (-\lambda^2 + \delta\lambda)(2\lambda - \delta)g(X, X) + (2\lambda - \delta)(g(\delta\phi^2 X, AX) + g(A\phi X, \phi AX)) \\ &= (-\lambda^2 + \delta\lambda)(2\lambda - \delta)g(X, X) \\ &\qquad + (2\lambda - \delta)(g(\delta(-X + \eta(X)\xi), AX) + g((\delta\lambda + (c/2))\phi X, \phi AX) \\ &= -\lambda^2(2\lambda - \delta)g(X, X) - (\delta\lambda + (c/2))g(\phi^2 X, AX) \\ &= -\lambda(2\lambda^2 - 2\delta\lambda - (c/2))g(X, X), \end{aligned}$$

so that $\lambda(2\lambda^2 - 2\delta\lambda - (c/2)) = 0$. We here note that $\lambda = 0$ is *not* a solution to the quadratic equation $2\lambda^2 - 2\delta\lambda - (c/2) = 0$. Hence our Hopf hypersurface M in $\widetilde{M}_n(c)$ has at most four constant principal curvatures δ, $\lambda = 0$ and λ_1, λ_2 which are solutions to the equation $2\lambda^2 - 2\delta\lambda - (c/2) = 0$. However we emphasize $\lambda \neq 0$ (see the tables of principal curvatures in Theorem 1, 2). So we see that our Hopf hypersurface M has at most three constant principal curvatures δ and λ_1, λ_2 which are solutions to the following quadratic equation:

$$2\lambda^2 - 2\delta\lambda - (c/2) = 0. \tag{7.3}$$

When our Hopf hypersurface M with constant principal curvatures does not have a principal curvature λ with $2\lambda - \delta = 0$, Equation (7.3) can be rewritten as:

$$\lambda = \frac{\delta\lambda + (c/2)}{2\lambda - \delta}.$$

This means $\phi AX = A\phi X$ for every X in V_λ, which, together with the fact that $\phi A\xi = 0 = A\phi\xi$, implies $\phi A = A\phi$ on M. Then M is locally congruent to a hypersurface of type (A) in a nonflat complex space form (see Proposition 3).

Finally, we investigate the case that our Hopf hypersurface M with constant principal curvatures has a principal curvature λ with $2\lambda - \delta = 0$. Then M is nothing but the horosphere HS in $\mathbb{C}H^n(c)$ (see the table of principal curvatures in the case of $c < 0$ in Theorem 2), so that M is a member of hypersurfaces of type (A). Therefore we obtain the desired conclusion.

8. Proof of Theorem 5

We first show that Condition (2) implies Condition (1). We denote by $\xi_{m,r}$ and $\eta_{m,r}$ the characteristic vector and the contact form on our geodesic sphere $G_m(r)$ in a Kähler manifold M. By the assumption $\nabla_{\xi_{m,r}}\psi_{m,r} = 0$ we have

$$\eta_{m,r}(A_{m,r}\xi_{m,r})^2 - \eta_{m,r}(A_{m,r}^2\xi_{m,r}) = 0$$

which corresponds to Equation (7.2), so that the geodesic sphere $G_m(r)$ is a Hopf hypersurface in the Kähler manifold M.

Next, in Lemma 8 we choose w orthogonal to both v and Jv and we put $u = Jv$. Since u_r is a normal vector on $G_m(r)$ in M at $\exp_m(ru)$, the vector v_r is the characteristic vector of $G_m(r)$ at this point. It follows from the fact that our geodesic sphere $G_m(r)$ is a Hopf hypersurface in M and Equation (6.1) that the curvature tensor R of M satisfies

$$g(R(u, Ju)w, u) = 0$$

(cf. [2]). Hence we can see that $R(u, Ju)u$ is proportional to Ju for every $u \in T_mM$, so that M is locally congruent to a complex space form (see [20]). Thus we obtain Condition (1).

Conversely, we suppose Condition (1). We take an arbitrary geodesic sphere $G(r)$ in a complex space form $M_n(c)$. Since $G_m(r)$ is a totally umbilic hypersurface in the case of $c = 0$, the tensor $\psi_{m,r} = \phi_{m,r}A_{m,r} - A_{m,r}\phi_{m,r}$ on $G_m(r)$ vanishes. On the other hand, when $c \neq 0$, our geodesic sphere $G_m(r)$ is not totally umbilic. However the tensor $\psi_{m,r}$ on $G_m(r)$ also vanishes (see Proposition 3). Therefore the tensor $\psi_{m,r}$ is parallel in a trivial sense. Thus we obtain Condition (2).

References

[1] T.Adachi, M.Kimura, S.Maeda, *A characterization of all homogeneous real hypersurfaces in a complex projective space by observing the extrinsic shape of geodesics*, Arch.Math. (Bssel) **73** (1999), 303–310.

[2] T.Adachi, S.Maeda, *Space forms from the viewpoint of their geodesic spheres*, Bull. Austral. Math. Soc. **62** (2000), 205–210.
[3] T. Adachi, S. Maeda and Y.H. Kim, *A characterization of the homogeneous minimal ruled real hypersurface in a complex hyperbolic space*, J. Math. Soc. Japan **61** (2009), 315–325.
[4] T. Adachi, S. Maeda and M. Yamagishi, *Length spectrum of geodesic spheres in a non-flat complex space form*, J. Math. Soc. Japan **54** (2002), 373–408.
[5] J. Berndt, *Real hypersurfaces with constant principal curvatures in complex hyperbolic space*, J. Reine Angew. Math. **395** (1989), 132–141.
[6] J. Berndt and H. Tamaru, *Cohomogeneity one actions on noncompact symmetric spaces of rank one*, Trans. Amer. Math. Soc. **359** (2007), 3425-3438.
[7] B.Y. Chen and S. Maeda, *Hopf hypersurfaces with constant principal curvatures in complex projective or complex hyperbolic spaces*, Tokyo J. Math. **24** (2001), 133–152.
[8] B. Y. Chen and L. Vanheche *Differential geometry of geodesic spheres*, J. Reine Angew. Math. **325** (1981), 28-67.
[9] T. Hamada, *Some real hypersurfaces of complex projective space*, Proc. of the Workshop on Differential Geometry of Submanifolds and its Related Topics. Saga, August 4-6, 2012, 82-86, World Scientific.
[10] T. Hamada, K. Yamashita, *The parallelism of a certain tensor of real hypersurfaces in a nonflat complex space form*, preprint
[11] M. Kimura, *Real hypersurfaces and complex submanifolds in complex projective space*, Trans. Amer. Math. Soc. **296** (1986), 137–149.
[12] S. Maeda, *A characterization of the homogeneous real hypersurfaces of type* (B) *with two distinct constant principal curvatures in a complex hyperbolic space*, Sci. Math. Japonicae **68** (2008), 677–686.
[13] S. Maeda, *Geometry of the horosphere in a complex hyperbolic space*, Differential Geometry and its Applications **29** (2011), S246–S250.
[14] S.Maeda,T.Adachi and Y.H.Kim, *Characterizations of geodesic hypersheres in a non-flat complex space form*, Glasgow Math.J. **55** (2013), 217–227
[15] Y. Maeda, *On real hypersurfaces of a complex projective space*, J. Math. Soc. Japan **28** (1976) 529-540.
[16] M. Okumura, *On some real hypersurfaces of a complex projective space*, Trans. Amer. Math. Soc. **212** (1975), 355-364.
[17] R. Niebergall and P.J. Ryan, *Real hypersurfaces in complex space forms*, Tight and Taut submanifolds, T.E. Cecil and S.S. Chern (eds.), Cambridge Univ. Press, 1998, 233–305.
[18] R.Takagi, *On homogeneous real hypersurfaces in a complex projective space*, Osaka J. Math. **10** (1973), 495–506.
[19] R.Takagi *Real hypersurfaces in a complex space with constant principal curvature*, J. Math. Soc. Japan **27** (1975), 43–53.
[20] S. Tanno, *Constancy of holomorphic sectional curvature in almost Hermitian manifolds*, Kodai Math. Sem. Rep. **25** (1973), 190-201.
[21] K. Yamashita and S. Maeda, *The real hypersurface of type (B) with two distinct principal curvatures in a complex hyperbolic space*, Kodai Math. J. **37** (2014), 24–33.

Department of Mathematics, Saga University, Saga 840-8502, Japan
E-mail address: `ky-karatsucity@sgr.bbiq.jp`